Optimization, integrals, and elasticity

1. Given the following utility function: $U = x^2 + y^3 + xy$ and the budget constraint $65 = 5x + 10y$
   (a) Construct the Lagrangian and calculate the first-order conditions.
   (b) Show that the first-order conditions imply that the marginal rate of substitution (or marginal rate of substitution) is equal to the price ratio. Approximately show this on a contour curve graph with an indifference curve concave to the origin, remember that the quantities of goods are positive.
   (c) Obtain the optimal quantities of x and y that maximize utility.

2. Given the following inverse demand function:

$$P = 10/Q + 1$$

   And the following supply function:

$$P = 2Q$$

   (a) Find the price elasticity of demand at the equilibrium point. Classify the elasticity.
   (b) Set up the double integral to calculate the producer surplus (the area that is below the price and above the supply curve). Suggestion: graph before setting up the integral.
   (c) Solve the double integral.

Solution

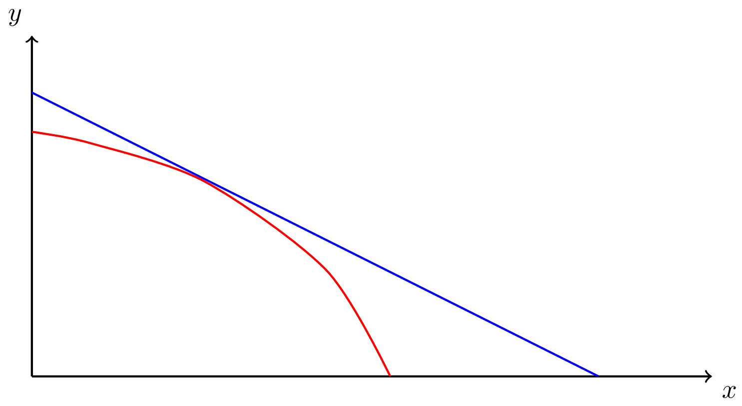
1. (a)

$$L = x^2 + y^3 + xy + \lambda(65 - 5x - 10y)$$
$$L'x = 2x + y - 5\lambda = 0$$
$$L'y = 3y^2 + x - 10\lambda = 0$$
$$L'\lambda = 65 - 5x - 10y = 0$$

(b) From the first two equations, I solve for λ and equate them:

$$\frac{2x+y}{5} = \frac{3y^2+x}{10}$$
$$\frac{2x+y}{3y^2+x} = \frac{5}{10}$$

The left-hand side is the MRS and the right-hand side is the price ratio.



(c) Solve for one variable in terms of the other:

$$2x + y = \frac{1}{2}(3y^2 + x)$$
$$\frac{3x}{2} = \frac{3y^2}{2} - y$$
$$x = y^2 - \frac{2y}{3}$$

Insert in the third condition:

$$25 - 5(y^2 - 2y/3) - 10y = 0$$
$$25 - 5y^2 + 10y/3 - 10y = 0$$
$$-5y^2 - 20y/3 + 65 = 0$$

Obtain two solutions $y = -4.33$ and $y = 3$. Discard the negative solution and replace $y = 3$ in the x function:

$$x = 3^2 - \frac{2}{3}(3)$$
$$x = 7$$

Therefore, the optimal quantities are $x = 7$ and $y = 3$

2. (a) Find the equilibrium by equating supply and demand:

$$10/Q + 1 = 2Q$$

$$Q = 2.5$$

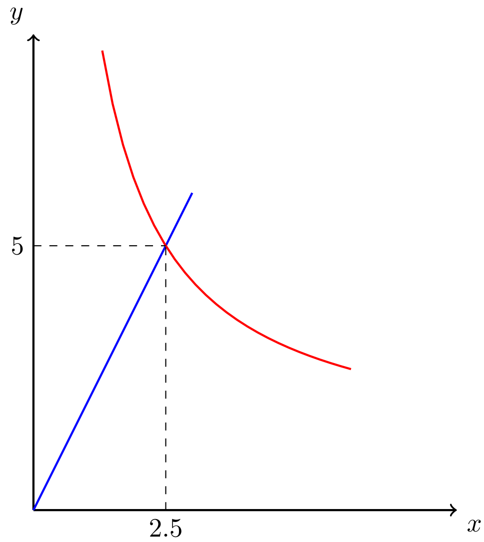
And the equilibrium price: $P = 5$. Calculate the price elasticity of demand, first solve for Q in terms of P:

$$Q = \frac{10}{P-1} = 10(P-1)^{-1}$$

$$\frac{\partial Q}{\partial P}\frac{P}{Q} = -10(P-1)^{-2}\frac{5}{2.5} = -10(5-1)^{-2}\frac{5}{2.5} = -1.25$$

It is elastic since $|-1.25| > 1$

(b) Graphing:



We set up the following 2 possible double integrals:

$$\int_0^{2.5}\int_{2x}^{5} dydx = \int_0^{5}\int_0^{y/2} dxdy$$

(c) Solve:

$$\int_0^{2.5}\int_{2x}^{5} dydx = \int_0^{2.5} 5 - 2xdx$$

Solving the integral:

$$5x - x^2$$

Evaluating at the limits:

$$5 * 2.5 - (2.5)^2 = 6.25$$